A NOTE ON THE PAPER: ON ITERATIONS FOR FAMILIES OF ASYMPTOTICALLY PSEUDOCONTRACTIVE MAPPINGS.

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Abstract. It is our aim in this note to give a counter example to an argument used in the proof of the main theorem of the paper: On iterations for families of asymptotically pseudocontractive mappings, *Applied Mathematics Letters,* **24** (2011), 33-38 by A. Rafiq [4]; and give an alternative condition to correct the anomaly.

1. Introduction.

This work is motivated by the recent paper of A. Rafiq [4]. Careful reading of Rafiq's work shows that there is a serious gap in the proof of Theorem 5 of [4], which happens to be main theorem of the paper.

It is our aim to give a counter example to the argument used in the proof of Theorem 5 of [4] and suggest an alternative condition in order to close the observed gap.

2. Preliminary.

Let E be a real Banach space with dual E^* and let $\langle ., \ . \rangle$ be the duality pairing between members of E and E^*. The mapping $J : E \to 2^{E^*}$ defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = ||x||^2; ||f^*|| = ||x||\}, x \in E,$$

is called the normalized duality mapping. We note that in a Hilbert space H, J is the identity operator. The single valued normalized duality mapping is denoted by j.

A mapping $T : D(T) \subset E \to E$ is said to be *L-Lipschitzian* if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\| \ \forall \ x, y \in D(T);$$

and T is said to be *uniformly L-Lipschitzian* if there exists $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\| \ \forall \ x, y \in D(T), \forall \ \ n \geq 1,$$

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where $D(T)$ denotes the domain of T. It is well known that the class of uniformly L-Lipschitzian mappings is *a proper subclass* of the class of L-Lipschitzian mappings.

The mapping T is said to be *asymptotically pseudocontractive* if there exists a sequence $\{k_n\}_{n\geq 1} \subset [1, +\infty)$ with $\lim_{n\to\infty} k_n = 1$ and for all $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that

$$\Big\langle T^n x - T^n y, j(x-y)\Big\rangle \leq k_n \|x-y\|^2 \ \forall\, x, y \in D(T),\ \forall\, n \geq 1.$$

In [4], A. Rafiq studied the strong convergence of the sequence $\{x_n\}_{n\geq 1}$ defined by

$$\begin{array}{rcl} & & x_1 \in K, \\ x_{n+1} & = & (1-\alpha_n)x_n + \alpha_n T_1^n y_n^1 \\ y_n^i & = & (1-\beta_n^i)x_n + \beta_n^i T_{i+1}^n y_n^{i+1} \\ & \vdots & \\ y_n^{p-1} & = & (1-\beta_n^{p-1})x_n + \beta_n^{p-1} T_p^n x_n,\ n \geq 1, \end{array} \tag{1.4}$$

for approximation of common fixed point of finite family of asymptotically pseudocontractive mappings in real Banach space. He proved the following theorem.

Theorem 2.1. *(See Theorem 5 of* [4]*) Let K be a nonempty closed convex subset of a real Banach space E and $T_l : K \to K,\ l = 1, 2, ..., p;\ p \geq 2$ be p asymptotically pseudocontractive mappings with T_1 and T_2 having bounded ranges and a sequence $\{k_n\}_{n\geq 1} \subset [1, +\infty),\ \lim_{n\to\infty} k_n = 1$ such that $x^* \in \bigcap_{l=1}^{p} F(T_l) = \{x \in K : T_1 x = x = T_2 x = ... = T_p x\}$. Further, let T_1 be uniformly continuous and $\{\alpha_n\}_{n\geq 1}$, $\{\beta_n^i\}_{n\geq 1}$, $\{\beta_n^{p-1}\}_{n\geq 1}$ be sequences in $[0, 1],\ i = 1, 2, ..., p;\ p \geq 2$ such that*
(i) $\lim_{n\to\infty} \alpha_n = 0 = \lim_{n\to\infty} \beta_n^1$;
(ii) $\sum_{n\geq 1} \alpha_n = \infty$.
For arbitrary $x_1 \in K$, let $\{x_n\}_{n\geq 1}$ be iteratively defined by (1.4)*. Suppose that for any $x^* \in \bigcap_{l=1}^{p} F(T_l)$, there exists a strictly increasing function $\Psi : [0, +\infty) \to [0, +\infty),\ \Psi(0) = 0$ such that*

$$(*) \quad \Big\langle T_l^n x - x^*, j(x-x^*)\Big\rangle \leq k_n \|x_n - x^*\|^2 - \Psi(\|x-x^*\|),\ \text{for all } x \in K,\ l = 1, 2, ..., p;\ p \geq 2.$$

Then $\{x_n\}_{n\geq 1}$ converges strongly to $x^* \in \bigcap_{l=1}^{p} F(T_l)$.

Remark 2.2. There are a lot to say about this result but let us first and formost address the major issue arising from the proof of this theorem.

On page 37 of [4], immediately after inequality (2.7), the author wrote:
“ From the condition (i) and (2.7), we obtain

$$\lim_{n\to\infty} \|y_n^1 - x_{n+1}\| = 0,$$

and the uniform continuity of T_1 leads to

$$\lim_{n\to\infty} \|T_1^n y_n^1 - T_1^n x_{n+1}\| = 0.\text{''}$$

This claim is, however, not true. To see this, we consider the following example:

Example 2.3. Let $\mathbb{R}$ denote the set of real numbers endowed with usual topology. Define $T : \mathbb{R} \to \mathbb{R}$ by $Tx = 2x \; \forall \; x \in \mathbb{R}$, then

$$|Tx - Ty| = 2|x - y| \;\; \forall \; x, y \in \mathbb{R}.$$

This implies that T is a Lipschitz mapping with Lipschitz constant $L = 2$. Thus, T is uniformly continuous since every Lipschitz map is uniformly continuous. Now, suppose $y_n^1 = 1 + \frac{1}{n}$ and $x_{n+1} = 1 - \frac{1}{n}$ for all $n \geq 1$, then

$$|y_n^1 - x_{n+1}| = \left|\Big(1 + \frac{1}{n}\Big) - \Big(1 - \frac{1}{n}\Big)\right| = \frac{2}{n} \to 0 \text{ as } n \to \infty.$$

We now show that

$$\lim_{n\to\infty} |T^n y_n^1 - T^n x_{n+1}| \neq 0.$$

Observe that

$$\begin{aligned}
Ty_n^1 &= 2y_n^1 = 2\Big(1 + \frac{1}{n}\Big) = 2 + \frac{2}{n} \\
T^2 y_n^1 &= T(Ty_n^1) = 2\Big(2 + \frac{2}{n}\Big) = 2^2 + \frac{2^2}{n} \\
T^3 y_n^1 &= T(T^2 y_n^1) = 2\Big(2^2 + \frac{2^2}{n}\Big) = 2^3 + \frac{2^3}{n} \\
&\vdots \\
T^n y_n^1 &= 2^n + \frac{2^n}{n} \text{ for all } n \geq 1.
\end{aligned}$$

Similar computation gives

$$T^n x_{n+1} = 2^n - \frac{2^n}{n} \text{ for all } n \geq 1.$$

Thus,

$$|T^n y_n^1 - T^n x_{n+1}| = \left|\Big(2^n + \frac{2^n}{n}\Big) - \Big(2^n - \frac{2^n}{n}\Big)\right| = \frac{2^{n+1}}{n} \; \forall \; n \geq 1.$$

It is easy to see (using mathematical induction) that $2^{n+1} \geq n \; \forall \; n \geq 1$. So,

$$|T^n y_n^1 - T^n x_{n+1}| = \frac{2^{n+1}}{n} \geq 1 \; \forall \; n \geq 1.$$

Hence,

$$\lim_{n\to\infty} |T^n y_n^1 - T^n x_{n+1}| \neq 0.$$

This contradicts the claim of A. Rafiq [4].

To correct the error in the result of A. Rafiq, we shall rather assume that T_1 is uniformly L-Lipschitzian so that

$$d_n = M\|T_1^n y_n^1 - T_1^n x_{n+1}\| \leq ML\|y_n^1 - x_{n+1}\| \to 0 \text{ as } n \to \infty.$$

The rest of the result follows as in [4].

Remark 2.4. In as much as the error in the proof of Theorem 5 of [4] has been pointed out and corrected, it is not clear what the author really want to achieve by constructing such a complicated scheme given by (1.4). If a clear study of the proof of [4] is made, one will easily observe that the mappings $T_l,\ 3 \leq l \leq p$ played no role at all. This suggests that the scheme will only make sense if only two operators T_1 and T_2 are considered. Besides, it is not specified in Theorem 5 of [4] which of the operators the sequence $\{k_n\}_{n\geq 1}$ is associated with. Meanwhile, condition $(*)$ gaurantees that the fixed point x^* of these operators is unique. This thus reduces the entire problem to what has been studied in [1] and [3]. We note that the result of Chidume and Chidume [2] and Ofoedu [3] remain correct if it were further assumed that the mapping $\phi : [0, +\infty) \to [0, +\infty)$ in thier results is onto.

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